J. Fluid Mech. (1967), *vol.* 29, *part* 2, *pp.* 241–257
Printed in Great Britain

Thermally induced boundary-layer flows in a rotating environment

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(Received 11 November 1966)

We consider, in this paper, boundary-layer flows which are induced when an isothermal rigid-body rotation is disturbed by heating the fluid. The basic rigid-body rotation is sustained by one or more rotating planes at which the temperature differences are initiated. Conditions of both uniform and unsteady heating are discussed.

1. Introduction

In this paper we consider fluid motions which are induced by heating fluid which is initially in a rigid-body rotation. We consider situations in which the fluid, rotating with angular velocity Ω, is bounded by plane surfaces having the same angular velocity. This isothermal rigid-body rotation will be disturbed if the temperature of the boundary is changed, since changes in temperature will be accompanied by changes in density which will modify the effect of the pressure gradient and a radial flow develops. We confine our attention to fluids of small viscosity and consider only those cases in which the disturbance is effectively confined to the neighbourhood of the plate. Such flows will be described by the boundary-layer equations which may be further simplified if we assume a linear variation of viscosity with temperature. If T_∞ is the temperature of the fluid in the isothermal rigid-body rotation and T_w the wall temperature we shall be interested in cases where $|T_w - T_\infty| \gg L^2\Omega^2/C_p$, $L^2\Omega^2/C_p \ll 1$ where L is a typical length and C_p the specific heat of the fluid. With this assumption we may neglect dissipative effects in the energy equation. Carrier (1966) has briefly considered the case when $|T_w - T_\infty| = O(L^2\Omega^2/C_p) \ll 1$ whilst Duncan (1966), using the Boussinesq approximation, has discussed flows driven by buoyancy forces when T_w is non-uniform. We discuss separately three different cases.

The first and simplest case concerns fluid bounded by the plane $z = 0$. For time $t < 0$ we have a rigid-body rotation with angular velocity Ω and at time $t = 0$ the temperature of the boundary is changed to a new uniform value T_w. Initially diffusion in the neighbourhood of the wall is the dominant process. However, as the fluid density changes a radial motion develops. If $T_w > T_\infty$ the less dense rotating fluid can no longer withstand the radial pressure gradient and moves radially inwards. As required by continuity an axial flow out from the boundary layer also begins to take place. Conversely, when the wall is cooled the heavier fluid in the neighbourhood of the rotating plane overcomes the radial pressure gradient and begins to move outwards. This fluid is replaced by an axial flow into the boundary layer from the main body of fluid. These processes continue. Thus for the heated wall the combined effects of convection and diffusion of heat away from the wall mean that a steady state in which the disturbance is confined to the neighbourhood of the wall is not realized. However, for the cooled wall there is a balance between the convection and diffusion processes resulting in a local steady-state distortion of the original rigid-body rotation. This case is examined in some detail. When $(T_\infty - T_w)/T_\infty = \epsilon \ll 1$ it is shown that the radial motion is confined to a region of thickness $O(\nu_\infty/\Omega)^{\frac{1}{2}}$, where ν is the kinematic viscosity, whereas the adjustment in temperature and angular velocity takes place over a region of thickness $O[\epsilon^{-1}(\nu_\infty/\Omega)^{\frac{1}{2}}]$. For $(T_\infty - T_w)/T_\infty = O(1)$ these two layers become indistinguishable with the disturbance extending over a region of thickness $O(\nu_\infty/\Omega)^{\frac{1}{2}}$.

As a second example we consider a solid body rotation between two planes, a distance $2L$ apart, each rotating with angular velocity Ω which is disturbed, at $t = 0$, by changing the temperature of each plate by the same amount, $O(\epsilon)$ where $\epsilon \ll 1$. As in the previous case we confine most of our remarks to the case where the planes are cooled. Greenspan & Howard (1963) have considered the analogous problem when, for isothermal flow, the angular velocities of the planes are each increased by the same small amount. Following the initial development of boundary layers at each wall, as described in the previous paragraph, a process corresponding to the 'spin-up' described by Greenspan & Howard may be identified. Fluid moving out radially in the boundary layers is replaced by fluid from the inviscid, isothermal interior. It is shown that in a time $O(R^{\frac{1}{2}}\Omega^{-1})$, where $R = \Omega L^2/\nu$ is a Reynolds number, the angular velocity of the interior motion increases by an amount $O(\epsilon)$ and the radial flow decays. Temperature variations are still confined, for sufficiently large values of R, to thin boundary layers adjacent to the walls and there follows a diffusive mode in which the new isothermal (with temperature T_w) rigid body rotation is established during a time $O(R\Omega^{-1})$. This contrasts with the work of Greenspan & Howard where the spin-up mode *effectively* establishes, in time $O(R^{\frac{1}{2}}\Omega^{-1})$, the new steady state, with small residual effects decaying under the action of viscous forces in time $O(R\Omega^{-1})$.

As a final example we consider an oscillatory flow induced when the temperature of the plane bounding an isothermal rigid body rotation assumes the value given by $(T_w - T_\infty) = \epsilon T_\infty \cos \omega t$ where $\epsilon \ll 1$. An analogous motion induced by a perturbation angular velocity $\epsilon\Omega \cos \omega t$ superimposed upon the basic rotation has been considered by Benney (1965). The fluctuating velocities and temperature $O(\epsilon)$ are confined to the familar Stokes 'shear-wave' layer of thickness $O(\nu_\infty/\omega)^{\frac{1}{2}}$ if $\omega < \Omega$. However, if $\omega > \Omega$ the disturbance extends over a region of thickness

$$O[(\nu_\infty/|\omega - 2\Omega|)^{\frac{1}{2}}].$$

When $\omega = 2\Omega$ a resonance phenomenon analogous to that exposed by Benney occurs and although the temperature fluctuations are confined to a region of thickness $O(\nu_\infty/\omega)^{\frac{1}{2}}$, the perturbation velocities penetrate into the interior fluid and we no longer have a local boundary-layer phenomenon. As is usual in fluctuating viscous flows of small amplitude (see Stuart 1963) steady components of velocity and temperature $O(\epsilon^2)$ are present. Benney shows that in the case of a flow induced by an angular velocity perturbation there is, at the edge of the boundary layer, a steady suction towards the plane if $\omega/\Omega \gg 1$ but if $\omega/\Omega \ll 1$ this steady flow is directed away from the plane. In the present case we show that at the edge of the boundary layer there is a rise in temperature $O(\epsilon^2 T_\infty)$ which in turn leads to a steady flow away from the plane for all values of ω/Ω. Thus, only to $O(\epsilon)$ do we have a local phenomenon. Heat is convected from the boundary with velocity $O[\epsilon^2(\nu_\infty \Omega)^{\frac{1}{2}}]$ into the interior and the boundary conditions, as posed there, cannot be satisfied. We consider briefly a special case when the basic rotation is sustained not only by the plane $z = 0$ but also by a parallel, thermally insulated rotating plane at a finite distance from this.

2. The boundary-layer equations

The boundary-layer equations (momentum, energy, mass conservation and state) for unsteady, axi-symmetric flow of a viscous, heat conducting fluid of small viscosity over a plane surface may be written as

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}\right) = -\frac{\partial p}{\partial r}+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right), \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}\right) = \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right), \tag{2}$$

$$\frac{\partial p}{\partial z} = 0, \tag{3}$$

$$\rho\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right) = \frac{1}{\sigma}\frac{\partial}{\partial z}\left(\mu\frac{\partial T}{\partial z}\right), \tag{4}$$

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r u)+\frac{\partial}{\partial z}(\rho w) = 0, \tag{5}$$

$$p = \rho R T. \tag{6}$$

In these equations t represents time, r and z are measured radially over the plane and axially. The velocity components u, v and w are in the radial, azimuthal and axial directions respectively; p denotes pressure and T temperature. The density ρ and viscosity μ are functions of T and we shall assume that the fluid is a gas in which

$$\mu \propto T. \tag{7}$$

The Prandtl number σ is assumed to be constant and $O(1)$. In the energy equation (3) the terms representing work done by the pressure forces, and viscous dissipation have been neglected, thus $|T_w - T_\infty| \gg L^2\Omega^2/C_p$. The quantity C_p is assumed to be constant.

In the problems considered in §§ 3–5 the boundary $z = 0$ has a constant angular velocity Ω about the axis $r = 0$. With the assumption (7) it is possible to simplify (1)–(6) as follows. We first note that if the fluid at a large distance from the plane also has uniform angular velocity Ω then we may write, by virtue of (3),

$\partial p/\partial r = \rho_\infty r\Omega^2$ where the subscript ∞ denotes conditions in the interior of the fluid away from the boundary. We now introduce a co-ordinate Z defined by

$$Z = \int_0^z \frac{\rho}{\rho_\infty} dz, \tag{8}$$

and a stream function ψ such that

$$u = \frac{1}{r}\frac{\rho_\infty}{\rho}\frac{\partial \psi}{\partial z} = \frac{1}{r}\frac{\partial \psi}{\partial Z}. \tag{9}$$

From the equation of continuity (5) we then have

$$\frac{\rho}{\rho_\infty} w = -\left\{\frac{\partial Z}{\partial t} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial Z}\left(\frac{\partial Z}{\partial r}\right)_{z,t}\right\}. \tag{10}$$

Thus, using (6), (7), (8), (9) and (10) together with $\rho\mu = \text{const.} = \rho_\infty \mu_\infty$, a consequence of (7), the momentum and energy equations (1)–(3) may be written as

$$\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial \psi}{\partial Z}\right) + \frac{1}{r}\frac{\partial \psi}{\partial Z}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial Z}\right) - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial}{\partial Z}\left(\frac{1}{r}\frac{\partial \psi}{\partial Z}\right) - \frac{v^2}{r} = -\frac{T}{T_\infty} r\Omega^2 + \nu_\infty \frac{\partial^2}{\partial Z^2}\left(\frac{1}{r}\frac{\partial \psi}{\partial Z}\right), \tag{11}$$

$$\frac{\partial v}{\partial t} + \frac{1}{r}\frac{\partial \psi}{\partial Z}\frac{\partial v}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial v}{\partial Z} + \frac{1}{r^2}\frac{\partial \psi}{\partial Z} v = \nu_\infty \frac{\partial^2 v}{\partial Z^2}, \tag{12}$$

$$\frac{\partial T}{\partial t} + \frac{1}{r}\frac{\partial \psi}{\partial Z}\frac{\partial T}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial T}{\partial Z} = \frac{\nu_\infty}{\sigma}\frac{\partial^2 T}{\partial Z^2}, \tag{13}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. We note in (11) that the radial pressure gradient, represented by the first term on the right-hand side, is enhanced by the factor T/T_∞. If the flow is isothermal with $T = T_\infty$ everywhere then a solution of (11)–(13) is

$$u = w = 0, \quad v = r\Omega. \tag{14}$$

However, if we have a situation in which $T \neq T_\infty$ everywhere, as when for example the wall temperature T_w is different from T_∞, then the balance between centrifugal force and pressure force is destroyed with a consequent departure from the rigid-body rotation (14).

In the examples which we consider below we are concerned, as already indicated, with plane surfaces perpendicular to which the axis of rotation of planes and fluid lies. It proves convenient to reduce equations (11)–(13) further by writing

$$\left.\begin{aligned} \psi &= r^2\Omega(2\nu_\infty/\Omega)^{\frac{1}{2}} AF(\eta,\tau), \\ v &= r\Omega\{1 + AG(\eta,\tau)\}, \\ T &= T_\infty\{1 + A\theta(\eta,\tau)\}, \\ \eta &= (\Omega/2\nu_\infty)^{\frac{1}{2}} Z, \\ \tau &= \Omega t, \end{aligned}\right\} \tag{15}$$

where A is a parameter as yet unspecified but related to the temperature, T_w, of the plane. In terms of these new variables the momentum and energy equations become

$$(\partial F'/\partial \tau) - 2G + A(F'^2 - 2FF'' - G^2) = -\theta + \tfrac{1}{2}F''', \tag{16}$$

$$(\partial G/\partial \tau) + 2F' + 2A(F'G - FG') = \tfrac{1}{2}G'', \tag{17}$$

$$(\partial \theta/\partial \tau) - 2AF\theta' = \frac{1}{2\sigma}\theta'', \tag{18}$$

where the primes denote differentiation with respect to η. At the solid boundary $\eta = 0$ we shall require

$$F(0,\tau) = F'(0,\tau) = G(0,\tau) = 0. \tag{19}$$

Other boundary conditions will be discussed in relation to the particular problem under consideration.

3. Uniformly heated plane

In this section we consider the disturbance to an isothermal solid body rotation induced when the temperature of the plane $z = 0$ which bounds the fluid is changed, at $\tau = 0$, from T_∞ to $T_\infty(1 \pm A)$ where $A > 0$. We examine separately the initial development and the steady flow which is subsequently realized. The boundary conditions which supplement (19) are

$$\left.\begin{aligned} F = G = \theta = 0 \quad (\tau = 0, \eta > 0),\\ \theta(0,\tau) = \pm 1 \quad (\tau > 0),\\ F'(\infty,\tau) = G(\infty,\tau) = \theta(\infty,\tau) = 0 \quad (\tau > 0).\end{aligned}\right\} \tag{20}$$

For the initial development it is convenient to use independent variables (ζ, τ) where

$$\zeta = \eta(\sigma/2\tau)^{\frac{1}{2}}, \tag{21}$$

and to expand F, G and θ as

$$\left.\begin{aligned} F &= \tau^{\frac{3}{2}} F_0(\zeta) + O(\tau^{\frac{7}{2}}),\\ G &= \tau^2 G_0(\zeta) + O(\tau^4),\\ \theta &= \theta_0(\zeta) + O(\tau^2).\end{aligned}\right\} \tag{22}$$

We shall consider only the lowest order terms θ_0 and F_0 which satisfy

$$\left.\begin{aligned} \theta_0'' + 2\zeta\theta_0' = 0,\\ \theta_0(0) = \pm 1, \quad \theta_0(\infty) = 0,\end{aligned}\right\} \tag{23}$$

and

$$\left.\begin{aligned} \sigma F_0'''' + 2\zeta F_0''' - 4F_0' = 4(2/\sigma)^{\frac{1}{2}}\theta_0,\\ F_0(0) = F_0'(0) = F_0'(\infty) = 0,\end{aligned}\right\} \tag{24}$$

where the primes now denote differentiation with respect to ζ.

The solution of (23), which represents pure diffusion, is

$$\theta_0 = \pm \operatorname{erfc}\zeta. \tag{25}$$

Equation (24) now gives, for F_0',

$$F_0' = \mp \left(\frac{2}{\sigma}\right)^{\frac{1}{2}} \left[\operatorname{erfc}\zeta + \frac{2}{(\sigma-1)\sqrt{\pi}}\left\{\sigma\left(1+\frac{2\zeta^2}{\sigma}\right)\frac{\sqrt{\pi}}{2}\operatorname{erf}\zeta + \zeta e^{-\zeta^2}\right\} - \left(1+\frac{2\zeta^2}{\sigma}\right) - \frac{2}{(\sigma-1)\sqrt{\pi}}\left\{\left(1+\frac{2}{\sigma}\zeta^2\right)\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{\zeta}{\sigma^{\frac{1}{2}}}\right) + \frac{\zeta}{\sigma^{\frac{1}{2}}}e^{-\zeta^2/\sigma}\right\}\right]. \tag{26}$$

One of the more interesting parameters from our point of view is $F_0(\infty)$ which is given from (26) as

$$F_0(\infty) = \mp \frac{2\sqrt{2}}{3(\pi\sigma)^{\frac{1}{2}}(\sigma^{\frac{1}{2}}+1)}. \tag{27}$$

Thus, initially, when the temperature of the boundary is changed we have a diffusive process described by (25). As the temperature of the fluid changes so does its density and the fluid then begins to move radially as the balance which resulted in solid body rotation is destroyed. We see from (27) that, for all $\sigma, F_0(\infty) < 0$ when $\theta_0(0) = 1$ and $F_0(\infty) > 0$ when $\theta_0(0) = -1$. Thus when the wall is heated there is an outflow from the boundary layer as fluid flows in radially along the wall. Conversely, when the wall is cooled the heavier fluid near the wall flows out radially and is replaced from the main body of fluid. This is as we expect.

With the main features of the flow for $\tau \ll 1$ exposed we turn now to the steady state which will be achieved when $\tau \gg 1$. We first consider the case when the temperature change is small and set $A = \epsilon \ll 1$. The steady state equations we must study are then given from (16)–(18) as

$$-2G + \epsilon(F'^2 - 2FF'' - G^2) = -\theta + \tfrac{1}{2}F'''', \tag{28}$$

$$2F' + 2\epsilon(F'G - FG') = \tfrac{1}{2}G'', \tag{29}$$

$$-2\sigma\epsilon F\theta' = \tfrac{1}{2}\theta'', \tag{30}$$

together with boundary conditions (19) and those of (20) which are appropriate for large τ. The wall temperature is given by $T_w = T_\infty(1 \pm \epsilon)$. Since $\epsilon \ll 1$, we seek a solution of (28)–(30) in the form $F(\eta, \epsilon) = \sum_{n=0} \epsilon^n F_n(\eta)$, etc. The first-order terms satisfy

$$\left.\begin{aligned} -2G_0 &= -\theta_0 + \tfrac{1}{2}F_0'''', \\ 2F_0' &= \tfrac{1}{2}G_0'', \\ \theta_0'' &= 0, \end{aligned}\right\} \tag{31}$$

with

$$\left.\begin{aligned} F_0(0) = F_0'(0) = G_0(0) = 0, \quad \theta_0(0) = \pm 1, \\ F_0'(\infty) = G_0(\infty) = \theta_0(\infty) = 0. \end{aligned}\right\} \tag{32}$$

The solution of (31) satisfying the conditions at $\eta = 0$ is

$$F_0 = \pm \frac{1-i}{8\sqrt{2}}\{-(1+i) + i\,e^{-(1+i)\sqrt{2}\,\eta} + e^{-(1-i)\sqrt{2}\,\eta}\}, \tag{33}$$

$$G_0 = \pm\{\tfrac{1}{2} - \tfrac{1}{4}(e^{-(1+i)\sqrt{2}\,\eta} + e^{-(1-i)\sqrt{2}\,\eta})\}, \tag{34}$$

$$\theta_0 = \pm 1. \tag{35}$$

The required solution is represented by the real parts of these expressions. We see that (34) and (35) cannot satisfy the conditions imposed at $\eta = \infty$. Consider first the case where the wall is heated. Equation (33) shows that there is outflow from the boundary layer at the wall and (34), (35) show that outside the boundary layer the fluid temperature is increased so that $T = T_\infty(1+\epsilon)$ everywhere, with a corresponding increase in the angular velocity necessary to maintain the rigid-body rotation. Clearly in this case diffusion of heat and vorticity from the wall will be reinforced by convection and the disturbance cannot be confined to the neighbourhood of the wall. The initial development of this procedure was observed in equations (25)–(27).

We now restrict our attention to the case when the wall is cooled. Equations (34) and (35) still indicate a change of $O(\epsilon)$ in the temperature and angular velocity everywhere outside the boundary layer. However, equation (33) shows now that $F_0(\infty) > 0$ and there is a net flux of fluid into the boundary layer replacing the fluid which is flowing out radially over the plane. Thus we may expect a balance between convection and diffusion with the disturbance confined to the neighbourhood of the boundary. We see from (30) that the expansion procedure adopted has reduced in importance the term which represents convection towards the wall. The solution (33)–(35) may thus only be considered as an inner solution and there must be an outer region in which there is a balance between convection and diffusion. We must therefore develop a solution in this outer region which is complementary to, and matches with, the inner solution whose first term is given by (33)–(35). In this outer region then, which is seen to be of thickness $O(\epsilon^{-1})$ times the thickness of the inner boundary layer, we set

$$\left.\begin{aligned} F = f(\zeta), \quad G = g(\zeta), \quad \theta = \phi(\zeta), \\ \text{where} \qquad \zeta = \epsilon\eta. \end{aligned}\right\} \tag{36}$$

We seek a solution, as for the inner solution, of the form $f(\zeta,\epsilon) = \sum_{n=0} \epsilon^n f_n(\zeta)$, etc., of the equations satisfied by f, g and ϕ namely

$$\left.\begin{aligned} -2g-\epsilon g^2+\epsilon^3(f'^2-2ff'') &= -\phi+\tfrac{1}{2}\epsilon^3 f''', \\ 2f'+2\epsilon(f'g-fg') &= \tfrac{1}{2}\epsilon g'', \\ -2\sigma f\phi' &= \tfrac{1}{2}\phi'', \end{aligned}\right\} \tag{37}$$

with boundary conditions

$$f'(\infty) = g(\infty) = \phi(\infty) = 0, \tag{38}$$

together with the condition that the solution of (37) should match, as $\zeta \to 0$, with the solution of (28)–(30) as $\eta \to \infty$. The first-order terms satisfy, from (37),

$$\left.\begin{aligned} 2g_0 &= \phi_0, \\ f_0' &= 0, \\ \phi_0''+4\sigma f_0\phi_0' &= 0. \end{aligned}\right\} \tag{39}$$

The second of these gives, by matching with (33),

$$f_0 = 1/4\sqrt{2}, \tag{40}$$

and the solution for ϕ_0, which matches with (35), is

$$\phi_0 = -e^{-\sigma\zeta/\sqrt{2}}. \tag{41}$$

The solution is completed by calculating g_0 from the first of equations (39). This process, developing the inner solution governed by (28)–(30) and the outer solution by (37), may be continued with unknown constants being determined by matching as above. Thus the terms $O(\epsilon)$ in (28)–(30) yield, as equations for $F_1(\eta)$, $G_1(\eta)$ and $\theta_1(\eta)$

$$\left.\begin{aligned} -2G_1+(F_0'^2-2F_0F_0''-G_0^2) &= -\theta_1+\tfrac{1}{2}F_1''', \\ 2F_1'+2(F_0'G_0-F_0G_0') &= \tfrac{1}{2}G_1'', \\ \theta_1'' &= 0, \end{aligned}\right\} \tag{42}$$

with

$$F_1(0) = F_1'(0) = G_1(0) = \theta_1(0) = 0, \tag{43}$$

and with the outer boundary conditions (20) replaced by the matching condition referred to above. Solutions of (42) are

$$\begin{aligned} F_1 = \left\{\frac{4i-3}{80\sqrt{2}}\right\}&\{-(1+i)+i\,e^{-(1+i)\sqrt{2}\,\eta}+e^{-(1-i)\sqrt{2}\,\eta}\} \\ &+\frac{\sqrt{2}\,i}{160}\{1-(1+i)\,e^{-(1+i)\sqrt{2}\,\eta}+i\,e^{-2\sqrt{2}\,\eta}\} \\ &+\frac{1}{8\sqrt{2}}\left\{\frac{(1-i)}{2\sqrt{2}}\,\eta\,e^{-(1+i)\sqrt{2}\,\eta}-\frac{i}{4}e^{-(1+i)\sqrt{2}\,\eta}\right\} \\ &+\frac{1}{8\sqrt{2}}\left\{\frac{(1+i)}{2\sqrt{2}}\,\eta\,e^{-(1-i)\sqrt{2}\,\eta}+\frac{i}{4}e^{-(1-i)\sqrt{2}\,\eta}\right\}, \end{aligned} \tag{44}$$

$$\theta_1 = a\eta. \tag{45}$$

The constant a, determined by matching with ϕ_0, is given by

$$a = \sigma/\sqrt{2}. \tag{46}$$

The function G_1 may be determined from the first of equations (42), in particular we have

$$G_1'(0) = \frac{1}{10\sqrt{2}}(5\sigma-1). \tag{47}$$

From (37) we see that the next term in the outer solution is given from

$$\left.\begin{aligned} -2g_1^2-g_0^2 &= -\phi_1, \\ 2f_1'+2(f_0'g_0-f_0g_0') &= \tfrac{1}{2}g_0'', \\ -2\sigma(f_0\phi_1'+f_1\phi_0') &= \tfrac{1}{2}\phi_1'', \end{aligned}\right\} \tag{48}$$

with boundary conditions $f_1'(\infty) = g_1(\infty) = \phi_1(\infty) = 0$ together with the matching requirement. Substituting for the first-order functions f_0, g_0 and ϕ_0 we see from the second of (48) that f_1 is constant, its value is determined from matching with (44) as

$$f_1 = 7/80\sqrt{2}. \tag{49}$$

The function ϕ_1 is now seen to satisfy

$$\phi_1''+\frac{\sigma}{\sqrt{2}}\phi_1' = -\frac{7\sigma^2}{40}e^{-\sigma\zeta/\sqrt{2}},$$

with $\phi_1(\infty) = 0$ giving

$$\phi_1 = C\,e^{-\sigma\zeta/\sqrt{2}}+\frac{7\sigma\sqrt{2}}{40}\zeta\,e^{-\sigma\zeta/\sqrt{2}}. \tag{50}$$

We see from the inner solution $O(\epsilon)$ that the matching condition requires $\phi_1(0) = 0$ and hence $C = 0$ giving

$$\phi_1 = \tfrac{7}{40}\sigma\sqrt{2}\zeta\,e^{-\sigma\zeta/\sqrt{2}}. \tag{51}$$

The solution to this order may be completed by calculating g_1 from the first of equations (48). The solution is not carried beyond this stage.

For small values of $A = (T_\infty-T_w)/T_\infty$ we see that the characteristic features of the flow are as follows. An inner boundary layer of thickness $O(\nu_\infty/\Omega)^{\frac{1}{2}}$, to which, effectively, the radial flow is confined and the temperature has the uniform value T_w, is embedded within a boundary layer of thickness $O[\epsilon^{-1}(\nu_\infty/\Omega)^{\frac{1}{2}}]$. In this outer region the temperature and angular velocity are adjusted to the values T_∞ and Ω respectively. The values of the shear stress and heat transfer may be calculated from $F''(0)$, $G'(0)$ and $\theta'(0)$ which are given, from (33), (34), (44), (45) and (47) as

$$\left.\begin{aligned} F''(0) &= 0{\cdot}7071 + 0{\cdot}0354\epsilon + O(\epsilon^2),\\ G'(0) &= -0{\cdot}7071 + (0{\cdot}3536\sigma - 0{\cdot}0707)\,\epsilon + O(\epsilon^2),\\ \theta'(0) &= 0{\cdot}7071\sigma\epsilon + O(\epsilon^2). \end{aligned}\right\} \tag{52}$$

For values of $A = O(1)$ we have used a Pohlhausen method to evaluate the main features of the flow. Equations (16)–(18), for steady flow, are integrated from $\eta = 0$ to $\eta = \infty$ to give

$$\left.\begin{aligned} -2\int_0^\infty G\,d\eta + A\int_0^\infty (3F'^2 - G^2)\,d\eta &= -\int_0^\infty \theta\,d\eta - \tfrac{1}{2}F''(0),\\ 2F(\infty) + 4A\int_0^\infty F'G\,d\eta &= -\tfrac{1}{2}G'(0),\\ 2\sigma A\int_0^\infty F'\theta\,d\eta &= -\tfrac{1}{2}\theta'(0). \end{aligned}\right\} \tag{53}$$

In conjunction with these equations we have assumed the following profiles, which satisfy the boundary conditions at the wall and at infinity,

$$\left.\begin{aligned} F &= \frac{1}{\lambda\mu(\lambda^2+\mu^2)}\{-\mu - \lambda\, e^{\lambda\eta}\sin\mu\eta + \mu\, e^{\lambda\eta}\cos\mu\eta\},\\ G &= -\tfrac{1}{2}e^{\delta\eta} + (1/4\lambda\mu)\{(\lambda^2-\mu^2)\,e^{\lambda\eta}\sin\mu\eta + 2\lambda\mu\, e^{\lambda\eta}\cos\mu\eta\},\\ \theta &= -e^{\delta\eta}. \end{aligned}\right\} \tag{54}$$

The parameters λ, μ and δ are to be determined from (53). The profiles (54) satisfy the linearized momentum equations for large η and reduce, effectively, to the correct forms when $A \ll 1$. However, when $A \ll 1$ we have seen that two length scales are involved and we cannot expect the Pohlhausen method outlined above to succeed. Consequently we have interpolated between the results of the approximate calculation and the results for $A \ll 1$. In order to carry out this interpolation satisfactorily the calculation was extended to values of A outside the range of interest ($A \leqslant 1$). The results of such a calculation for $\sigma = 1$ are shown in figures 1–4. These show, in effect, a gradual merging of the two layers discussed earlier together with a thinning of the total region to which the disturbance to the isothermal rigid-body rotation is confined. This latter result is shown in figure 4 which is a measure of the thickness of this region in terms of the length scale $(\nu_\infty/\Omega)^{\frac{1}{2}}$.

4. A 'spin-up' phenomenon

In this section we consider the disturbance to the isothermal rigid-body rotation between two parallel rotating planes a distance $2L$ apart when the temperature of each plane is changed from T_∞ to T_w at time $t = 0$. We shall show that the final steady state is again an isothermal rigid-body rotation but with temperature T_w.

As in the previous example we pay most attention to the case in which the walls are cooled and restrict ourselves to the case $(T_\infty - T_w)/T_\infty \ll 1$. Thus at $t = 0$ the temperature of the walls is changed to $T_\infty(1-\epsilon)$. Initially boundary layers of the type discussed in § 3 develop on each plane. Thus radial outflow is confined to a

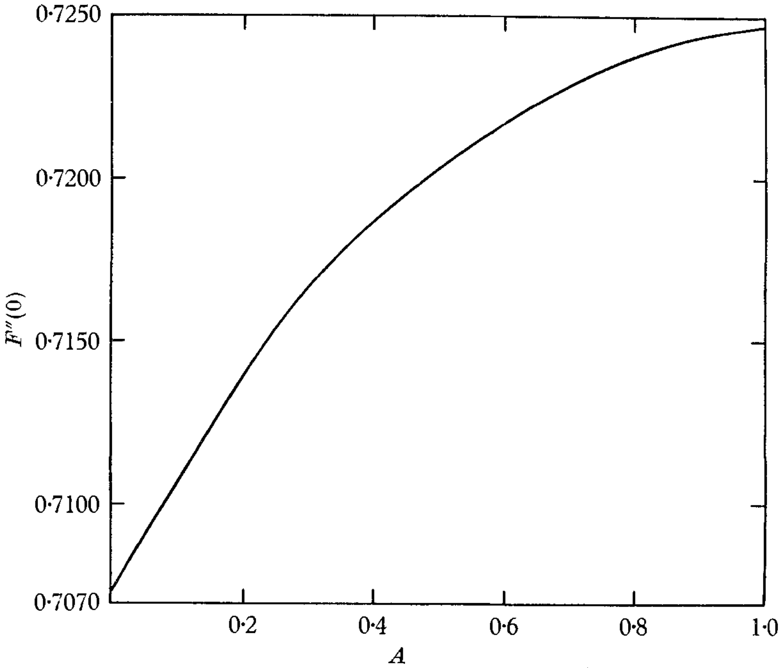

FIGURE 1. The dependence of $F''(0)$ on wall temperature when $\sigma = 1$.

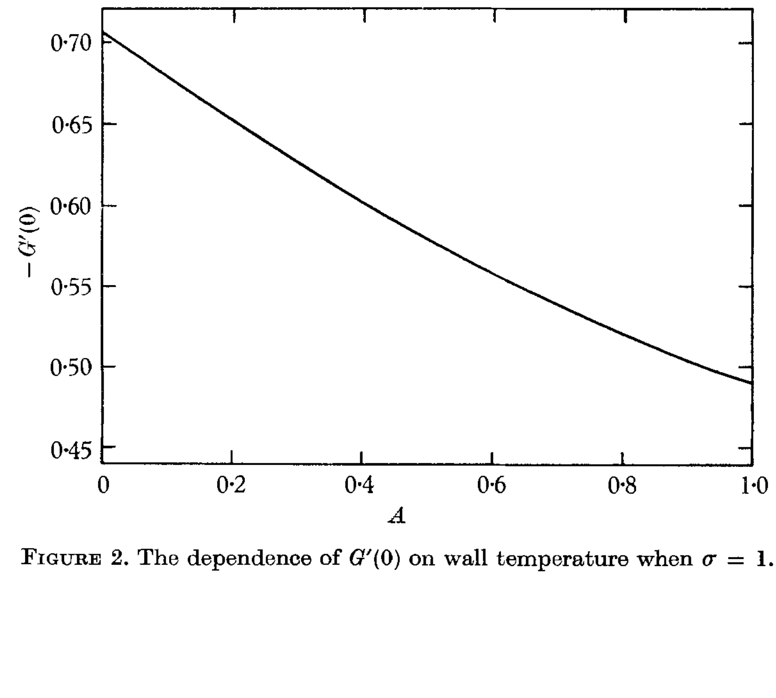

FIGURE 2. The dependence of $G'(0)$ on wall temperature when $\sigma = 1$.

boundary layer of thickness $O(R^{-\frac{1}{2}}L)$ whilst temperature variations extend over a region of thickness $O(\epsilon^{-1}R^{-\frac{1}{2}}L)$. Here $R = \Omega L^2/\nu_\infty$ is a Reynolds or Taylor number which is assumed to be sufficiently large so that $\epsilon^2 R \gg 1$. The radial velocity in the inner boundary layer and inflow velocity to the boundary layer are $O(\epsilon\Omega L)$ and $O(\epsilon\Omega L R^{-\frac{1}{2}})$ respectively. Continuity then requires that, in the

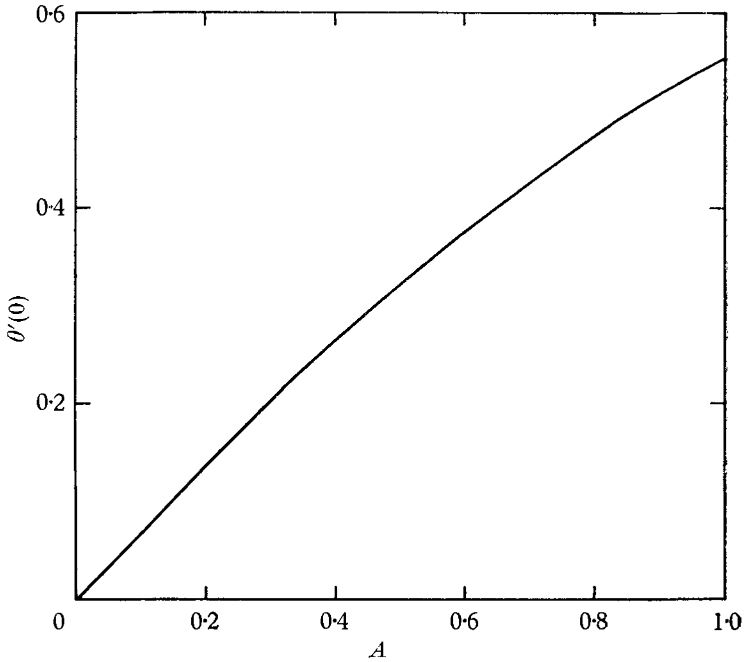

FIGURE 3. The dependence of $\theta'(0)$ on wall temperature when $\sigma = 1$.

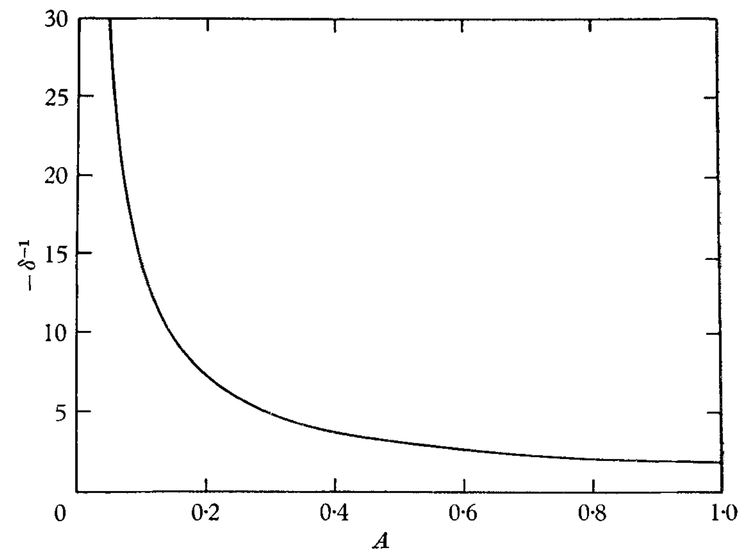

FIGURE 4. The boundary-layer thickness, on the scale $(\nu_\infty/\Omega)^{\frac{1}{2}}$, as a function of A when $\sigma = 1$.

inviscid, isothermal interior region the fluid has a radial inward component of velocity $O(\epsilon\Omega LR^{-\frac{1}{2}})$. Thus an annular ring of fluid moves radially inwards a distance $O(\epsilon L)$ during a time $O(R^{\frac{1}{2}}\Omega^{-1})$ and since in the inviscid interior angular momentum is conserved, the angular velocity of the fluid must increase by an amount $O(\epsilon\Omega)$. In the problem considered by Greenspan & Howard (1963), where the angular velocity of the planes was increased by an amount $O(\epsilon\Omega)$, the 'spin-up' process described above was the main agency by which a new rigid-body rotation with angular velocity $\Omega(1+\epsilon)$ was achieved. Small residual effects were found to decay under the action of viscosity in a time $O(R\Omega^{-1})$ but the principal features of the new steady state appeared in a time $O(R^{\frac{1}{2}}\Omega^{-1})$. As we shall see below, for the present problem the main features of the new steady state are not a consequence of the spin-up mode but are established by a diffusive process in time $O(R\Omega^{-1})$.

Consider first the motion in the inviscid interior. With L as reference length $\bar{r}$ and $\bar{z}$ are dimensionless cylindrical polar co-ordinates measured from the axis of rotation and the plane of symmetry so that the rotating planes lie on $\bar{z} = \pm 1$. We introduce a stream function ψ_I defined by

$$\psi_I = \epsilon R^{-\frac{1}{2}} L^3 \Omega \chi_I, \tag{55}$$

so that the continuity equation (5) (with $\rho = \rho_\infty$) is satisfied by

$$\left.\begin{aligned} u &= \epsilon L\Omega R^{-\frac{1}{2}} \frac{1}{\bar{r}} \frac{\partial \chi_I}{\partial \bar{z}}, \\ w &= -\epsilon L\Omega R^{-\frac{1}{2}} \frac{1}{\bar{r}} \frac{\partial \chi_I}{\partial \bar{r}}. \end{aligned}\right\} \tag{56}$$

We also write the azimuthal component of velocity v as

$$v = L\Omega\bar{r}(1+\epsilon G_I), \tag{57}$$

where G_I is to be independent of $\bar{r}$. Since we are interested in the spin-up phenomenon which takes place in a time $O(R^{\frac{1}{2}}\Omega^{-1})$ we choose as our dimensionless time, τ, defined by

$$t = R^{\frac{1}{2}}\Omega^{-1}\tau. \tag{58}$$

Substituting (55)–(58) into the inviscid, incompressible equations of motion we have, to lowest order,

$$(1+2\epsilon G_I) = \frac{1}{\bar{r}}\frac{\partial \bar{p}}{\partial \bar{r}}, \tag{59}$$

$$\bar{r}^2 \frac{\partial G_I}{\partial \tau} + 2\frac{\partial \chi_I}{\partial \bar{z}} = 0, \tag{60}$$

$$\partial \bar{p}/\partial \bar{z} = 0, \tag{61}$$

where $\bar{p} = \rho_\infty L^2\Omega^2 p$. Equations (59) and (60) then show that

$$\partial G_I/\partial \bar{z} = 0,$$

or

$$G_I = G_I(\tau). \tag{62}$$

Equation (60) then gives

$$\chi_I = -\tfrac{1}{2}\bar{z}\bar{r}^2(\partial G_I/\partial \tau) + \chi_0(\bar{r},\tau), \tag{63}$$

where χ_0 is an arbitrary function of $\bar{r}$ and τ. The solution, to this order, is completed following a discussion of the boundary-layer solution. We see from (59) that the varying interior angular velocity implies that a time-dependent radial pressure gradient is imposed upon the boundary layer.

We turn now to the boundary-layer solution and for the lower plane we write

$$\bar{z} = -1 + (2/R)^{\frac{1}{2}}\eta, \tag{64}$$

where η is as defined in (15). With the stream function associated with the disturbance velocities (9) and (10), the azimuthal velocity and the temperature given by

$$\left.\begin{aligned} \psi &= \epsilon\Omega L^3(2/R)^{\frac{1}{2}}\bar{r}^2F(\eta,\tau), \\ v &= \Omega L\bar{r}\{1+\epsilon G(\eta,\tau)\}, \\ T &= T_\infty\{1+\epsilon\theta(\eta,\tau)\}, \end{aligned}\right\} \tag{65}$$

essentially as in (15), and with the dimensionless time given by (58) the boundary-layer equations (1)–(4) become, using (59) and retaining only the leading terms,

$$2(G_I - G) + \theta = \tfrac{1}{2}F''', \quad 4F' = G'', \quad \theta'' + 4\epsilon\sigma F\theta' = 0, \tag{66, 67, 68}$$

together with (19) and

$$\theta(0,\tau) = -1, \quad F'(\infty,\tau) = \theta(\infty,\tau) = 0, \quad G(\infty,\tau) = G_I. \tag{69}$$

The term $O(\epsilon)$ in (68) has been retained in anticipation of the difficulty which occurred in § 3. The boundary condition imposed on F' at infinity recognizes the fact that the boundary layer and interior radial velocities differ by a factor $O(R^{-\frac{1}{2}})$. The quantity τ now only appears in (66)–(69) as a parameter.

The first-order solution of (66)–(68), corresponding to (33)–(35), may be written as

$$\left.\begin{aligned} F &= -\tfrac{1}{2}\gamma(1-i)\{-(1+i) + ie^{-(1+i)\sqrt{2}\eta} + e^{-(1-i)\sqrt{2}\eta}\}, \\ G &= -\tfrac{1}{2} + G_I + \sqrt{2}\,\gamma\{e^{-(1+i)\sqrt{2}\eta} + e^{-(1-i)\sqrt{2}\eta}\}, \\ \theta &= -1, \end{aligned}\right\} \tag{70}$$

where
$$\gamma(\tau) = (1-2G_I)/4\sqrt{2}. \tag{71}$$

As in § 3 this solution, which does not satisfy the boundary conditions at infinity, may only be considered as an inner solution. The adjustment of angular velocity and temperature takes place in an outer boundary layer of thickness $O(\epsilon^{-1}R^{-\frac{1}{2}}L)$. In the outer layer we introduce the variables (36) and the leading terms satisfy

$$2(g - G_I) = \phi, \quad f' = 0, \quad \phi'' + 4\sigma f\phi' = 0, \tag{72}$$

with $f'(\infty,\tau) = \phi(\infty,\tau) = 0$, $g(\infty,\tau) = G_I$, together with the condition that the solution of (72) must match with the inner solution (70). The solutions of (72) are thus obtained as

$$f = \gamma(\tau), \quad g = G_I(\tau) - \tfrac{1}{2}e^{-4\sigma\gamma\zeta}, \quad \phi = -e^{-4\sigma\gamma\zeta}. \tag{73}$$

A boundary layer with similar characteristics develops on the plane $\bar{z} = 1$.

The solution is completed when we have determined $G_I(\tau)$. This we do by matching the axial velocities of the boundary layers and interior flows. Consider first the boundary layer associated with $\bar{z} = -1$. From (56) and (63), as $\bar{z} \to -1$,

$$w \sim -\epsilon L\Omega R^{-\frac{1}{2}}\left\{\frac{\partial G_I}{\partial \tau}+\frac{1}{\bar{r}}\frac{\partial \chi_0}{\partial \bar{r}}\right\},$$

whilst from (8), (10), (58), (65) and (73), as $\eta \to \infty$,

$$w \sim -2\sqrt{2}\,\epsilon L\Omega R^{-\frac{1}{2}}\gamma(\tau).$$

Consequently, if these two expressions are to agree

$$\frac{1-2G_I}{2} = \frac{\partial G_I}{\partial \tau}+\frac{1}{\bar{r}}\frac{\partial \chi_0}{\partial \bar{r}}. \tag{74}$$

This matching condition when applied to the boundary layer on $\bar{z} = 1$ gives

$$\frac{1-2G_I}{2} = \frac{\partial G_I}{\partial \tau}-\frac{1}{\bar{r}}\frac{\partial \chi_0}{\partial \bar{r}}. \tag{75}$$

Together (74) and (75) give, as the equation for G_I,

$$(\partial G_I/\partial \tau)+G_I = \tfrac{1}{2}, \tag{76}$$

and if we apply the condition that as $\tau \to 0$, $G_I \to 0$ so that solutions (70) and (73) match with (33)–(35), (40) and (41) then

$$G_I = \tfrac{1}{2}\{1-e^{-\tau}\}. \tag{77}$$

We see then, from equations (70)–(77), that after a time $O(R^{\frac{1}{2}}\Omega^{-1})$ the radial flow, which was initially established in boundary layers of thickness $O(R^{-\frac{1}{2}}L)$, decays. The flow pattern now consists essentially of an interior motion, occupying most of the flow field, in which we have an isothermal rigid-body rotation with temperature T_∞ and angular velocity $\Omega(1+\frac{1}{2}\epsilon)$ together with boundary layers of thickness $O(\epsilon^{-1}R^{-\frac{1}{2}}\gamma^{-1}L)$ at $\bar{z} = \pm 1$ (which remain thin as long as $\tau \ll \log \epsilon^2 R$) through which the temperature and angular velocity are adjusted to their boundary values T_w and Ω. The process by which a new isothermal steady state of solid body rotation with temperature T_w and angular velocity Ω is now established is one of diffusion from the boundaries. This diffusive process takes place in a time $O(R\Omega^{-1})$. In the problem considered by Greenspan & Howard the final steady state was effectively reached in a time $O(R^{\frac{1}{2}}\Omega^{-1})$ with small residual effects, not revealed by the present boundary-layer analysis, finally decaying under the action of viscous diffusion in a time $O(R\Omega^{-1})$. As indicated by Greenspan & Howard the above analysis will not require modification if vertical side walls are present at a radial distance D, except in the immediate neighbourhood of these walls, provided that $L/D \ll R^{\frac{1}{2}}$.

Suppose now that the wall temperatures are increased by an amount ϵT_∞ at $t = 0$ so that $T_w = T_\infty(1+\epsilon)$. We have seen in § 3 that heat is convected away from the boundaries into the interior by velocities $O(\epsilon R^{-\frac{1}{2}}L\Omega)$. Consequently during the spin-up mode the effects of wall heating will have penetrated to a depth $O(\epsilon L)$ from the boundaries. Thus again the final rigid-body rotation will be established in a time $O(R\Omega^{-1})$ by diffusive processes.

5. An oscillatory motion

We discuss finally an oscillatory flow which is superimposed upon the rigid-body rotation when the plane $z = 0$, in otherwise unbounded fluid, has a wall temperature

$$T_w = T_\infty(1+\epsilon\cos\omega t), \tag{78}$$

where $\epsilon \ll 1$. An oscillatory motion induced by superimposing a small harmonic oscillation upon the basic rotation of the plane has recently been considered by Benney (1965).

In this case there is a new viscous length scale $(\nu_\infty/\omega)^{\frac{1}{2}}$ in addition to the length scale $(\nu_\infty/\Omega)^{\frac{1}{2}}$. In the familiar examples of oscillatory viscous flows (see Stuart 1963) of small amplitude, the first-order fluctuations are confined to a shear-wave layer of thickness $O(\nu_\infty/\omega)^{\frac{1}{2}}$. Here we shall see that the disturbance may penetrate outside such a shear wave layer. In Benney's work neither of the above length scales appear explicitly.

We define our variables as in § 2 with $A = \epsilon$ except that in view of (78) we choose ω^{-1} as a reference time. The equations satisfied by F, G and θ are given by (16)–(18) except that the time derivative terms now have coefficient

$$\alpha = \omega/\Omega. \tag{79}$$

In order to minimize the manipulative details we consider only the case $\sigma = 1$. The boundary conditions are as in (19) and (20) except that we now require

$$\theta(0,\tau) = e^{i\tau}. \tag{80}$$

We again seek a perturbation solution as a series in powers of ϵ with the first terms satisfying

$$\left.\begin{aligned} \alpha\frac{\partial F_0'}{\partial\tau} - 2G_0 &= -\theta_0 + \tfrac{1}{2}F_0''', \\ \alpha\frac{\partial G_0}{\partial\tau} + 2F_0' &= \tfrac{1}{2}G_0'', \\ \alpha\frac{\partial\theta_0}{\partial\tau} &= \tfrac{1}{2}\theta_0''. \end{aligned}\right\} \tag{81}$$

With (80) in mind we seek solutions of equations (81) in the form

$$F_0(\eta,\tau) = F_{01}(\eta)\,e^{i\tau}, \quad \text{etc.}$$

The functions F_{01}, G_{01} and θ_{01} which satisfy the appropriate boundary conditions are found to be

$$\left.\begin{aligned} F_{01} &= \frac{i}{4m_2}(1-e^{-m_2\eta}) - \frac{i}{4m_1}(1-e^{-m_1\eta}), \\ G_{01} &= \tfrac{1}{2}e^{-(1+i)\sqrt{\alpha}\,\eta} - \tfrac{1}{4}(e^{-m_1\eta}+e^{-m_2\eta}), \\ \theta_{01} &= e^{-(1+i)\sqrt{\alpha}\,\eta}, \end{aligned}\right\} \tag{82}$$

where, with positive real parts, m_1 and m_2 are given from

$$m_1^2 = 2i(\alpha+2), \quad m_2^2 = 2i(\alpha-2). \tag{83}$$

We see, from (79), (82) and (83) that, regardless of the value of α, first-order temperature fluctuations are indeed confined to a shear-wave layer of thickness $O(\nu_\infty/\omega)^{\frac{1}{2}}$. If $\omega < \Omega$ the total disturbance is confined to this region. However, if $\omega > \Omega$ velocity fluctuations persist outside this layer and extend to a distance $O(\nu_\infty/|\omega-2\Omega|)^{\frac{1}{2}}$. If $\omega = 2\Omega$ we have the resonance phenomenon observed by Benney when velocity fluctuations are no longer confined to thin boundary layers adjacent to the plane $z = 0$.

We consider now second-order effects which contain not only fluctuating quantities of twice the applied oscillation frequency but also steady terms. The appearance of these steady terms is consistent with the general theory of oscillatory, small amplitude viscous flows (Stuart 1963). We thus write our dependent variables in the form

$$F_1(\eta,\tau) = F_{10}(\eta)+F_{12}(\eta)\,e^{2i\tau}, \quad \text{etc.} \tag{84}$$

Consider first the second-order solution θ_1 for the temperature which satisfies

$$\alpha(\partial\theta_1/\partial\tau)-\tfrac{1}{2}\theta_1'' = 2F_0\,\theta_0', \tag{85}$$

with

$$\theta_1(0) = \theta_1(\infty) = 0. \tag{86}$$

The term θ_{12} derived from this represents a decaying oscillation. However the steady part persists and the solution shows that at the edge of the boundary layer we have

$$\theta_{10}(\infty) = \begin{cases} \dfrac{\alpha^2}{2(\alpha^2-1)^2} & (\alpha > 2); \\ \dfrac{1}{4\alpha^{\frac{1}{2}}(2-\alpha)^{\frac{1}{2}}}-\dfrac{\alpha^{\frac{1}{2}}}{4(2-\alpha)^{\frac{1}{2}}[(2-\alpha)^{\frac{1}{2}}+\alpha^{\frac{1}{2}}]^2}-\dfrac{\alpha}{8(1+\alpha)^2} & (\alpha < 2). \end{cases} \tag{87}$$

There is a corresponding persistence of angular velocity. The calculation of F_1 is a formidable task and we confine our comments here to the value of $F_{10}(\infty)$. This quantity determines, in particular, the nature of the induced steady flow at the edge of the boundary layer. Benney in his work discussed three limiting cases and showed, in the present notation, that

$$F_{10}(\infty) \sim \begin{cases} -3\sqrt{2}/20 & \text{as} \quad \alpha \to 0; \\ -(29\sqrt{2}-30)/34 & \text{as} \quad \alpha \to 2; \\ \sqrt{2}/4\alpha & \text{as} \quad \alpha \to \infty. \end{cases} \tag{88}$$

Thus, for sufficiently large values of α there is a suction velocity at the edge of the boundary layer, otherwise there is an outflow from the boundary layer. We have considered in detail only the equivalent of the last of these which shows, that as $\alpha \to \infty$, $F_{10}(\infty) \sim -\sqrt{2}/16\alpha^2$. Thus in contrast to Benney's result we have, in this case, outflow from the boundary layer. That we may expect such an outflow from the boundary layer for all values of α may be inferred from (87). This equation shows that for all α, $\theta_{10}(\infty) > 0$ and so at the edge of the boundary layer a steady temperature in excess of T_∞ by an amount $O(\epsilon^2 T_\infty)$ persists. The work of § 3 shows that with such a rise in temperature we may associate an outflow from the boundary layer with outflow velocities $O[(\nu_\infty\,\Omega)^{\frac{1}{2}}\epsilon^2]$. Thus heat is convected into the interior and only to $O(\epsilon)$ is the disturbance confined to the neighbourhood of the plane $z = 0$.

Let us suppose that the basic rigid-body rotation involves a second plane rotating with angular velocity Ω at $z = L$. Let us also suppose that this plane is thermally insulated with an artificially applied suction velocity corresponding to $F_{01}(\infty)\, e^{i\tau}$. In this way we isolate, at the upper plane, the difficulty associated with the steady streaming by ensuring that there is no interaction between it and the oscillatory motion $O(\epsilon)$. It is clear from the work of § 3 that at the upper plane there will be an outer boundary layer of thickness $O[\epsilon^{-2}(\nu_\infty/\Omega)^{\frac{1}{2}}]$ through which the temperature and angular velocity change. Embedded within this is a layer of constant temperature of thickness $O(\nu_\infty/\Omega)^{\frac{1}{2}}$ in which there is a radial outflow and the velocities are adjusted to satisfy the no-slip condition on the plane $z = L$. At the plane $z = L$ a temperature $T_\infty(1 + O(\epsilon^2))$ will be recorded.

This work was performed with the sponsorship of the United States Air Force under Contract no. AF 49(638)-1274 whilst the author was visiting the Dept. of Aeronautics and Astronautics, Stanford University. The hospitality of that department is gratefully acknowledged.

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